Transition Constraints: A Study on the Computational Complexity of Qualitative Change

Matthias Westphal, Julien Hué, Stefan Wölfl, and Bernhard Nebel

Institut für Informatik
Albert-Ludwigs-Universität Freiburg

IJCAI 2013
Outline

Motivation

Preliminaries

Problem definition

Sequential CSPs

Conclusion
Qualitative spatial and temporal reasoning (QSTR) is a useful aspect in numerous fields (GIS, robot navigation, . . . )

Often QSTR describes static relations between entities . . .

“the coffee machine is to the right of the sink”
Motivation

- Qualitative spatial and temporal reasoning (QSTR) is a useful aspect in numerous fields (GIS, robot navigation, . . .)
- Often QSTR describes static relations between entities . . .
  “the coffee machine is to the right of the sink”
- . . . however, in practice, relations may change over time . . .
  “the coffee machine has been moved”
Motivation

- Qualitative spatial and temporal reasoning (QSTR) is a useful aspect in numerous fields (GIS, robot navigation, ...)

- Often QSTR describes static relations between entities . . .
  "the coffee machine is to the right of the sink"

- . . . however, in practice, relations may change over time . . .
  "the coffee machine has been moved"

- . . . and we assume no object teleports through space.
  "the coffee machine suddenly appeared in my office"

continuity of change
Qualitative Change in the Literature

- Numerous works in the QSTR community: Freksa, Galton, Gerevini, Nebel, Apt, Brand, . . .
  With different focuses
  - (formal) continuity
  - computational complexity
  - application framework

- Different concepts of transitions for qualitative relations
  Sometimes equivalent, sometimes not
Qualitative Change in the Literature

- Numerous works in the QSTR community: Freksa, Galton, Gerevini, Nebel, Apt, Brand, ... With different focuses
  - (formal) continuity
  - computational complexity
  - application framework
- Different concepts of transitions for qualitative relations
  Sometimes equivalent, sometimes not
- Here we propose transition relations
  Relations just like (static) qualitative relations
  ... allows for theoretic study and the same algorithms
Qualitative Change in the Literature (Cont’d)

Example from Cui et al., 1992

Qualitative simulation of cellular behavior.

- qualitative relations between regions
- interstate constraints govern transitions
- simple formalization of neighborhood graphs
Example from Westphal et al., 2011

- Manipulation task
- Pick up objects using gripper
- Typical stochastic RMP-approaches generate unnatural plans
Example from Westphal et al., 2011

- First generate a qualitative plan which guides the sampling of the stochastic RMP
- Set of possible world states
- Use search to find one path from start to finish
Example from Westphal et al., 2011

- Simple qualitative relations between rectangles
- Only the gripper moves
- Only polynomially many states
- Plans can be generated in polynomial time

- What about plan generation with multiple robot arms?
Qualitative Reasoning

Reasoning about qualitative aspects of time and/or space (infinite domains) with finitely many qualitative relations between entities of interest.

Basic Relations of the Point Algebra (PA)

<table>
<thead>
<tr>
<th>Relation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I &lt; J$</td>
<td>$I \quad J$</td>
</tr>
<tr>
<td>$I = J$</td>
<td>$I, J$</td>
</tr>
<tr>
<td>$I &gt; J$</td>
<td>$J \quad I$</td>
</tr>
</tbody>
</table>
Conceptual Neighborhood Graph (Freksa)

Figure: Neighborhood graph for the point algebra.

Example

\[ y \rightarrow z \]

\[ x \rightarrow y \rightarrow z \]

\[ - - - \rightarrow \]

\[ x \] jumps from \[ y \] to \[ z \] ignoring the intermediary inevitable state.
Directed Neighborhood Graph (Galton; Wölfl)

Figure: Directed Neighborhood Graph for the point algebra.

- Dotted edges: (point, interval)-transitions.
- Dashed edges: (interval, point)-transitions.
A relational signature \( \{R_1, \ldots, R_m\} \)

A constraint language is an interpretation of the signature over a domain \( D \), written \( \Gamma = \langle D; R_1, \ldots, R_m \rangle \)

Primitive-positive formulae are of the form
\[
\varphi = \exists v_1, \ldots, v_n \bigwedge_{j=1}^{l} \psi_j \text{ where } \psi_j \text{ is a relation } R_i(v_{j_1}, \ldots, v_{j_{k_i}}) \text{ or binary equality.}
\]

An instance of \( \text{CSP}(\Gamma) \) is a primitive-positive formula without free variables
The Point Algebra

- The **Point Algebra** $\Gamma^{PA}$ is given by the signature

$$\{\emptyset, <, \leq, \neq\}$$

with the usual interpretation on $\mathbb{Q}$.

- A **fragment** of the PA is any $\Gamma$ built on $S \subseteq \{\emptyset, <, \leq, \neq\}$ with the same interpretation of symbols as $\Gamma^{PA}$.

- The **basic Point Algebra** is given by the signature $\{<\}$ with the usual interpretation on $\mathbb{Q}$.

- Solving instances of CSP($\Gamma^{PA}$) is polynomial time!

- ... still, some fragments are “easier” than others.
Spatial Calculi Stemming from the PA

- Cardinal Direction
  - Freksa 1991
- Interval Algebra
  - Allen 1983
- Block Algebra
  - Balbiani 1998
- Rectangle Algebra
  - Guesgen 1989

- All four are strictly more expressive than the PA . . .
- . . . they “contain” the PA
Sequences of States

Definition

An instance of SeqCSP(Γ) is a tuple $S = \langle V, (Q^1, \ldots, Q^d) \rangle$, with $V$ a finite set of variables and $Q^1, \ldots, Q^d$ instances of CSP(Γ) over subsets of $V$.

- $Q^t$ is called a state
- $v^t$ refers to the variable $v$ in $Q^t$
- A base solution of $S$ is a sequence $\alpha = \alpha^1, \ldots, \alpha^d$ of solutions of $Q^1, \ldots, Q^d$ respectively.
\( T_2 \) Transition

**Definition**

\( \alpha \) is a \( T_2 \)-solution if it satisfies \( \forall 1 \leq i, j \leq n \) and \( \forall 1 \leq t < d \):

\[
\alpha^t(v_i) < \alpha^t(v_j) \Rightarrow \alpha^{t+1}(v_i) \leq \alpha^{t+1}(v_j),
\]

i.e., it respects the conceptual neighborhood graph.

**Example**

\[
\begin{align*}
Q^1 & \rightarrow Q^2 \\
Q^2 & \rightarrow Q^3 \\
Q^3 & \rightarrow Q^4
\end{align*}
\]
\( T_4 \) Transition

**Definition**

A \( T_2 \)-solution \( \alpha \) is a \( T_4 \)-solution if \( \forall 1 \leq i, j, k, l \leq n \) and \( \forall 1 \leq t < d \):

\[
\alpha^t(v_i) \neq \alpha^t(v_j) \land \alpha^t(v_k) = \alpha^t(v_l) \Rightarrow \\
\neg (\alpha^{t+1}(v_i) = \alpha^{t+1}(v_j) \land \alpha^{t+1}(v_k) \neq \alpha^{t+1}(v_l))
\]

i.e., it respects the directed neighborhood graph.

**Example (Not a \( T_4 \) Transition)**

\[
\begin{array}{cccc}
\text{x} & \text{y} & \text{z} & \text{a} & \text{b} & \text{c} \\
\end{array}
\]

\( Q^1 \)

\[
\begin{array}{cccc}
\text{x} & \text{y} & \text{z} & \text{a} & \text{b} & \text{c} \\
\end{array}
\]

\( Q^2 \)

- From intervals to points and points to intervals.
Based on $T_2$- and $T_4$-solutions we can define $T_2$ and $T_4$ relations:

**Definition**

\[
vw \ T_2 \ v'w' := \neg(v < w \land v' > w') \land \neg(v > w \land v' < w').
\]

\[
v_1 \ldots v_4 \ T_4 \ v'_1 \ldots v'_4 := \bigwedge_{1 \leq i < j \leq 4} v_i v_j \ T_2 \ v'_i v'_j \land \bigwedge_{1 \leq i, j, k, l \leq 4} ((v_i \neq v_j \land v_k = v_l) \rightarrow \neg(v'_i = v'_j \land v'_k \neq v'_l)).
\]

With the natural interpretation of $T_2$, $T_4$ on $\mathbb{Q}$, we have constraint languages: $\langle \mathbb{Q}; \emptyset, <, \leq, \neq, T_2 \rangle \langle \mathbb{Q}; \emptyset, <, \leq, \neq, T_4 \rangle$
Example

With \( V = \{v^1, v^2, w^1, w^2\} \), set as constraints
\[ \{v^1 < w^1, v^1 < v^2, w^1 = w^2, v^1 w^1 T_2 v^2 w^2\} \].

- \( v \) is initially left of \( w \) and gets closer to \( w \) in the second state
- \( w \) remains in the same position
Complexity of these Constraint Languages

Proposition (from Bodirsky and Kára)

All $\Gamma$ with first order-definable relations using $<$ over $\mathbb{Q}$ are in NP.

Corollary

Thus, for any set $S$ of PA symbols, CSP($\langle \mathbb{Q}; S, T_2 \rangle$) and CSP($\langle \mathbb{Q}; S, T_4 \rangle$) are in NP.

Proposition

For any set $S$ of PA symbols, that contains $\neq$ or $<$, CSP($\langle \mathbb{Q}; S, T_2 \rangle$) and CSP($\langle \mathbb{Q}; S, T_4 \rangle$) are NP-complete.

Proof idea: Reduction from Betweenness.
The Maximal Tractable Class

Proposition

For any $S \subseteq \{\emptyset, \leq\}$, both CSP($\langle Q; S, T_2 \rangle$) and CSP($\langle Q; S, T_4 \rangle$) are tractable.

Proof idea: If $\emptyset$ is present in an instance it is unsatisfiable. Otherwise, assign the same value to every variable

- Our study shows the largest tractable class is $S = \{\emptyset, \leq\}$
- How about tractability for SeqCSP?
NP-Completeness of SeqCSP with \{\neq\}

**Theorem**

\(\text{SeqCSP}(\langle Q; \neq \rangle; T_2)\) and \(\text{SeqCSP}(\langle Q; \neq \rangle; T_4)\) are NP-complete.

**Proof idea**

Reduction from Betweenness

\[\{x^1 = y^1, y^1 \neq z^1, x^2 \neq y^2, y^2 \neq z^2, x^2 \neq z^2, x^3 \neq y^3, y^3 = z^3\}\]

Satisfies \(x^2 < y^2 < z^2\) or \(x^2 > y^2 > z^2\).

\[Q^1\quad y \quad z \quad \rightarrow\]
\[Q^2\quad x \quad y \quad \rightarrow\]
\[Q^3\quad x \quad y \quad z \quad \rightarrow\]

\[\quad x \quad y \quad z \quad \rightarrow\]
\[\quad y \quad z \quad x \quad \rightarrow\]
NP-Completeness of SeqCSP with \( \{\neq\} \)

**Theorem**

\( \text{SeqCSP}(\langle Q; \neq\rangle; T_2) \) and \( \text{SeqCSP}(\langle Q; \neq\rangle; T_4) \) are NP-complete.

**Proof idea**

Reduction from Betweenness

\( \{x^1 = y^1, y^1 \neq z^1, x^2 \neq y^2, y^2 \neq z^2, x^2 \neq z^2, x^3 \neq y^3, y^3 = z^3\} \)

Satisfies \( x^2 < y^2 < z^2 \) or \( x^2 > y^2 > z^2 \).
NP-Completeness of SeqCSP with \( \neq \)

**Theorem**

\( \text{SeqCSP}(\langle Q; \neq \rangle; T_2) \) and \( \text{SeqCSP}(\langle Q; \neq \rangle; T_4) \) are NP-complete.

**Proof idea**

Reduction from Betweenness

\[ \{ x^1 = y^1, y^1 \neq z^1, x^2 \neq y^2, y^2 \neq z^2, x^2 \neq z^2, x^3 \neq y^3, y^3 = z^3 \} \]

Satisfies \( x^2 < y^2 < z^2 \) or \( x^2 > y^2 > z^2 \).
NP-Completeness of SeqCSP with \{<, \leq}\n
**Theorem**

\[
\text{SeqCSP}(\langle \mathbb{Q}; <, \leq; T_2 \rangle) \text{ is NP-complete.}
\]

**Proof idea**

Reduction from 1-IN-3-SAT:

- Introduce a variable \( v_a \) for each atom \( a \).
- Express \( \neq: \{ v_a^1 < t^1, L^1 = O^1 = R^1 \} \) and \( \{ v_a^2 = t^2, L^2 < O^2 < R^2 \} \) forces \( v_a^2 \neq O^2 \);
- Create a SeqCSP-instance where \( v_a^2 < L^2 \) is related to \( a \) is true and \( R^2 < v_a^2 \) to \( a \) is false.
- No two \( v_i \) in the same clause can ever be on the right-hand side.
NP-Completeness of SeqCSP with \{<, \leq\}

**Theorem**

\(\text{SeqCSP}(\langle \mathbb{Q}; <, \leq; T_2 \rangle)\) is NP-complete.

**Proof idea (Cont’d)**

For each pair of atoms \(a\) and \(b\) in the same clause.

\[
\begin{align*}
l_a^s &= l_{l_a}^s = a^s = r r_a^s = r_a^s \land l_b^s = b^s = r r_b^s < r_b^s \\
l_a^{s+1} < l_{l_a}^{s+1} = a^{s+1} = r r_a^{s+1} < r_a^{s+1} \land l_b^{s+1} = b^{s+1} = r r_b^{s+1} = r_b^{s+1} \\
l_a^{s+2} &= l_{l_a}^{s+2} < a^{s+2} = r r_a^{s+2} = r_a \land l_b^{s+2} < b^{s+2} = r r_b^{s+2} < r_b^{s+2}
\end{align*}
\]

To force the situation

\[
\begin{array}{cccc}
R & a & b \\
l_a & r_a & l_b & r_b \\
\end{array}
\]

And then

\[
l_a^{s+3} < r_a^{s+3} \land l_b^{s+3} < r_b^{s+3} \land a^{s+3} \leq R^{s+3} \land b^{s+3} \leq R^{s+3}
\]
NP-Completeness of SeqCSP with $\{<\}$

Complexity results can be strengthened:

**Theorem**

$\text{SeqCSP}(\langle \mathbb{Q}; < \rangle; T_2)$ and $\text{SeqCSP}(\langle \mathbb{Q}; < \rangle; T_4)$ are NP-complete.

Proof idea: similar to the previous one.
Overview of Results

\[
\langle Q; \emptyset, \leq, \neq \rangle; T \\
\langle Q; \emptyset, <, \leq, \neq \rangle; T \\
\langle Q; \leq \rangle; T \\
\langle Q; < \rangle; T \\
\emptyset, <, \leq, \neq, T \\
\emptyset, \leq, T \\
<, T \\
\neq, T \\
\emptyset, <, \leq, \neq
\]

P, NP-complete for \( T \) either \( T_2 \) or \( T_4 \).
Fixed-parameter Tractability

Proposition

SeqCSP(Γ^{PA}; T_2) and SeqCSP(Γ^{PA}; T_4) are fixed-parameter tractable in the number of variables |V|.

Proof.

▶ Treat every Q^t as a set of possible states (fixed size in |V|),
▶ connect states at neighboring time points if they satisfy the T_2- (or T_4-) condition,
▶ use graph search to find a path.
Conclusion and Perspectives

To summarize:

▶ qualitative spatio-temporal problems as constraint languages
▶ defined $T_4$-constraint for continuous transitions of points
▶ NP-completeness for (interesting) fragments with the PA
▶ results transfer to IA, BA, RA, CD, . . .

Future work:

▶ investigate similar problem for non point-based formalisms
▶ make use of tractability in implementations (CP/SAT)
Main references

- Cui, Cohn, and Randell - Qualitative simulation based on a logical formalism of space and time. *AAAI*. 1992

Thank you for your attention